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Strong mixing measures for C_0-semigroups

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Abstract Our purpose is to obtain a very effective and general method to prove that certain C_0-semigroups admit invariant strongly mixing measures. More precisely, we show that the frequent hypercyclicity criterion for C_0-semigroups ensures the existence of invariant strongly mixing measures with full support. We will provide several examples, that range from birth-and-death models to the Black–Scholes equation, which illustrate these results.

Keywords Semigroup of operators · Strongly mixing measure · Frequently hypercyclic

1 Introduction

The interest in the dynamics of C_0-semigroups of operators comes from the analysis of the asymptotic behaviour of solutions to certain linear partial differential equations and to infinite systems of linear differential equations. Especially, the chaotic behaviour (in the topological and in the measure-theoretic sense) of C_0-semigroups has experienced a great development in recent years (see, e.g., [1,8–10,16,18,20,30,31,38]).

We recall that $(T_t)_{t\geq 0}$, with $T_t : X \to X$ a continuous and linear map on a Banach space X for each $t \geq 0$, is a C_0-*semigroup* if $T_0 = I$, $T_{t+s} = T_t \circ T_s$ and $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$. If X is a separable infinite-dimensional Banach space, a C_0-*semigroup*

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$(T_t)_{t\geq 0}$ on X is said to be *hypercyclic* if there exists $x \in X$ such that the set $\{T_t x : t \geq 0\}$ is dense in X. An element $x \in X$ is a *periodic point* for the semigroup if there exists $t > 0$ such that $T_t x = x$. A semigroup $(T_t)_{t\geq 0}$ is called *Devaney chaotic* if it is hypercyclic and the set of periodic points is dense in X.

There are analogous properties related to C_0-semigroups defined on a probability space $(X, \mathfrak{B}, \mu)$, where X is a Banach space and $\mathfrak{B}$ denotes the σ-algebra of Borel subsets of X. We will say that a Borel probability measure μ has *full support* if for any non-empty open set $U \subset X$ we have $\mu(U) > 0$. A measure μ is said to be T_t*-invariant* if for all $A \in \mathfrak{B}$ we have that $\mu(A) = \mu(T_t^{-1}(A))$ for all $t \geq 0$. A C_0-semigroup is *strongly mixing* if

$$\lim_{t\to\infty} \mu\left(A \cap T_t^{-1}(B)\right) = \mu(A)\mu(B) \qquad (A, B \in \mathfrak{B}).$$

Strongly mixing implies *ergodicity*, i.e., for each $A \in \mathfrak{B}$ such that $T^{-1}(A) = A$, either $\mu(A) = 0$, or $\mu(A) = 1$.

The concept of frequent hypercyclicity was introduced by Bayart and Grivaux [12] inspired by Birkhoff's ergodic theorem. The first ones that used ergodic theory for the dynamics of linear operators were Rudnicki [37] and Flytzanis [25]. The notion of frequent hypercyclicity was extended to C_0-semigroups in [3]. We recall that the lower density of a measurable set $M \subset \mathbb{R}_+$ is defined by

$$\underline{Dens}(M) := \liminf_{N\to\infty} \frac{\lambda(M \bigcap [0, N])}{N},$$

where λ is the Lebesgue measure on $\mathbb{R}_+$. A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that $\underline{Dens}(\{t \in \mathbb{R}_+ \; ; \; T_t x \in U\}) > 0$ for any non-empty open set $U \subset X$.

The first version of a frequent hypercyclicity criterion for operators was obtained by Bayart and Grivaux [12]. Later, Bonilla and Grosse-Erdmann [17] gave a more general formulation for operators on separable F-spaces. Another (probabilistic) version of it was provided by Grivaux [27].

In [32], Mangino and Peris obtained a continuous version of the frequent hypercyclicity criterion based on the Pettis integral. This criterion can be verified in certain cases in terms of the infinitesimal generator of the semigroup. They also gave applications for C_0-semigroups generated by Ornstein–Uhlenbeck operators, and for translation semigroups on weighted spaces of p-integrable (or continuous) functions. Their main result was the following sufficient condition for frequent hypercyclicity.

Theorem 1 ([32]) *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If there exist $X_0 \subset X$ dense in X and maps $S_t : X_0 \to X$, $t > 0$, such that*

(i) $T_t S_t x = x, T_t S_r x = S_{r-t} x, t > 0, r > t > 0$ *for all* $x \in X_0$,
(ii) $t \to T_t x$ *is Pettis integrable in* $[0, \infty)$ *for all* $x \in X_0$,
(iii) $t \to S_t x$ *is Pettis integrable in* $[0, \infty)$ *for all* $x \in X_0$,

then $(T_t)_{t\geq 0}$ is frequently hypercyclic.

Our purpose is to show that this criterion suffices for the existence of invariant Borel probability measures on X that are strongly mixing and have full support.

This is a continuous version of a result that we obtained for single operators [33]. More precisely, under the hypothesis of Bonilla and Grosse-Erdmann [17], the authors derived a stronger result by showing that a T-invariant strongly mixing measure with full support can be obtained.

Theorem 2 ([33]) *Let T be an operator on a separable F-space X. If there are, a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$, $n \in \mathbb{N}$, such that, for each $x \in X_0$:*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally*
(ii) $\sum_{n=0}^{\infty} S_n x$ *converges unconditionally, and*
(iii) $T^n S_n x = x$ *and* $T^m S_n x = S_{n-m} x$ *if* $n > m$,

then there is a T-invariant strongly mixing Borel probability measure μ on X with full support.

In contrast with the chaotic behaviour in the topological sense, which is trivial to pass from the discrete to the continuous case, while difficult or false to go in the other direction (see, e.g., [19] for hypercyclicity and frequent hypercyclicity, and [11] for Devaney chaos), the measure-theoretic properties are not trivially passed from the discrete to the continuous case, especially because of the requirement of T_t-invariance for every $t > 0$. This is why we need to construct explicitly the strongly mixing measures for C_0-semigroups, and they cannot be obtained from the main result in [33].

Our notation is standard, and we refer to the recent books [13] and [28] for the basic theory on chaotic linear dynamics.

We also recall the main definitions and results about Pettis integrability that will be needed in the paper. The proofs of all these results can be found in [23] for the case of a finite measure space, but they easily extend to σ-finite measure spaces. Let X be a Banach space and (Ω, μ) a σ-finite measure space. A function $f : \Omega \to X$ is said to be *weakly μ-measurable* if the scalar function $\varphi \circ f$ is μ-measurable for every $\varphi \in X'$, where X' denotes the topological dual of X; f is said to be *μ-measurable* if there exists a sequence $(f_n)_n$ of simple functions such that $\lim_{n\to\infty} \|f_n - f\| = 0$ μ-a.e.

Dunford's lemma says that, if f is weakly μ-measurable and $\varphi \circ f \in L_1(\Omega, \mu)$ for every $\varphi \in X'$, then for every measurable $E \subseteq \Omega$ there exists $x_E \in X''$ such that

$$x_E(\varphi) = \int_E \varphi \circ f \, d\mu,$$

for every $\varphi \in X'$. When $f : \Omega \to X$ is weakly μ-measurable and $\varphi \circ f \in L_1(\Omega, \mu)$ for every $\varphi \in X'$, then f is called *Dunford integrable*. The Dunford integral of f over a measurable $E \subseteq \Omega$ is defined by the element $x_E \in X''$ such that $x_E(\varphi) = \int_E \varphi \circ f \, d\mu$, for every $\varphi \in X'$.

In the case that $x_E \in X$ for every measurable E, then f is said to be *Pettis integrable* and x_E is called the Pettis integral of f over E, which is denoted by $(P) - \int_E f \, d\mu$. Clearly the Dunford and Pettis integrals coincide if X is a reflexive space. Moreover, if $\|f\|$ is integrable on Ω (i.e. f is *Bochner integrable* on Ω), then f is Pettis integrable on Ω. A basic result of Pettis says that, if f is Pettis integrable, then for every sequence $(E_n)_n$ of disjoint measurable sets in Ω

$$\int_{\bigcup_{n\in\mathbb{N}} E_n} f \, d\mu = \sum_{n\in\mathbb{N}} \int_{E_n} f \, d\mu,$$

where the series converges unconditionally. As a consequence, if $f : [0, +\infty[\to X$ is Pettis integrable on $[0, +\infty[$, then for every $\varepsilon > 0$ there exists $N > 0$ such that for every compact set $K \subset [N, +\infty[$

$$\left\| \int_K f(t) \, dt \right\| < \varepsilon.$$

2 Invariant measures and the frequent hypercyclicity criterion

We are now ready to present our main result.

Theorem 3 *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If there exist, $X_0 \subset X$ dense in X and maps $S_t : X_0 \to X$, $t > 0$ such that:*

(i) $T_t S_t x = x, T_t S_r x = S_{r-t}x, t > 0, r > t > 0$ *for all* $x \in X_0$,
(ii) $t \to T_t x$ *is Pettis integrable in* $[0, \infty)$ *for all* $x \in X_0$,
(iii) $t \to S_t x$ *is Pettis integrable in* $[0, \infty)$ *for all* $x \in X_0$,

then there is a $(T_t)_{t\geq 0}$-invariant strongly mixing Borel probability measure μ on X with full support.

The idea behind the proof is to construct, given a C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Banach space X satisfying the hypothesis of Theorem 1,

1. a "model" probability space $(Z, \overline{\mu})$ and
2. a Borel measurable map $\Phi : Z \to X$ with dense range,

where

- $(R_t)_{t\in\mathbb{R}}$ is the translation group defined as $R_t f(x) = f(x - t)$,
- $Z \subset C(\mathbb{R})$ is a $(R_t)_{t\in\mathbb{R}}$-invariant subset of the space $C(\mathbb{R})$ of continuous functions on the real line, endowed with its natural Frchet space compact-open topology,
- $\overline{\mu}$ is a $(R_t)_{t\in\mathbb{R}}$-invariant strongly mixing measure with full support, and
- $\Phi R_t = T_t \Phi$ on Z for all $t \geq 0$.

As a consequence, the Borel probability measure μ on X defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is $(T_t)_{t\geq 0}$-invariant, strongly mixing, and has full support. Proving the measure μ is $(T_t)_{t\geq 0}$-invariant and strongly mixing is simple. Showing μ has full support will take a little work.

Proof We suppose $X_0 = \{x_n; n \in \mathbb{N}\}$ with $x_1 = 0$. Let $U_n = B(0, \frac{1}{2^n})$, the open ball of radius $1/2^n$ centered at 0. By conditions (ii) and (iii) we can obtain an increasing sequence $\{N_n\}_n \in \mathbb{N}$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that, for any sequence $(C_k)_k$ of mutually disjoint compact sets with $C_k \subset [k/2, +\infty[$, $k \in \mathbb{N}$, we have that

$$\text{if } m_k \leq 2l, \quad \text{for } N_l \leq k < N_{l+1}, \quad l \geq n, \ \ n \in \mathbb{N},$$

then

$$\sum_{k\geq N_n} \int_{C_k} T_t x_{m_k}\, dt \in U_{n+1} \ \text{ and } \ \sum_{k\geq N_n} \int_{C_k} S_t x_{m_k}\, dt \in U_{n+1}. \tag{1}$$

1. The model probability space $(Z, \overline{\mu})$.

First of all, we define the following set $A \subset C(\mathbb{R})$ of continuous functions: $f \in A$ if there exist a sequence $(s_i)_{i\in\mathbb{Z}}$ of real numbers such that

(a) $\ldots s_{-4} < s_{-2} < 0 \leq s_0 < s_2 < s_4 < \ldots$,
(b) $|s_{2i+2} - s_{2i} - 1| \leq \frac{1}{2}$, and
(c) $s_{2i+1} = (s_{2i} + s_{2i+2})/2$, $i \in \mathbb{Z}$;

and a sequence of natural numbers $(n_i)_{i\in\mathbb{Z}}$ such that

(d) $f(s_{2i}) = n_i$,

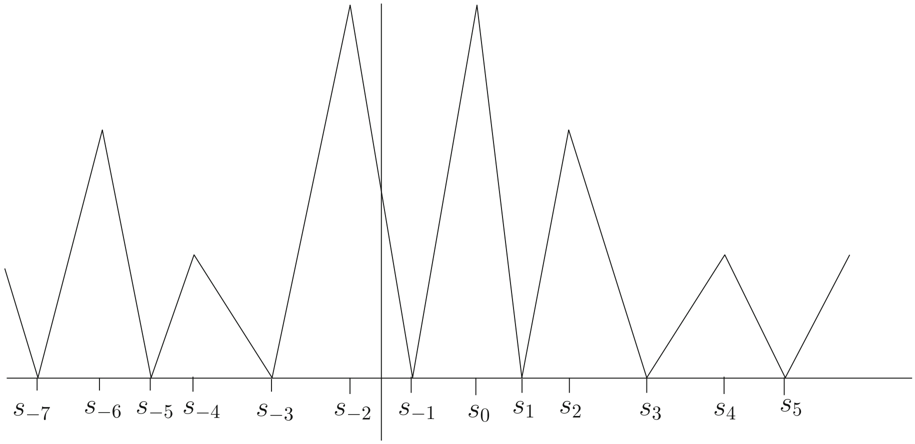

Fig. 1 Graph of a typical function $f \in A$

(e) $f(s_{2i+1}) = 0$, and
(f) $f''|_{]s_i, s_{i+1}[} \equiv 0$ for all $i \in \mathbb{Z}$.

We write $f_{(s_{2k}, n_k)_k}$ to denote the continuous function f associated with sequences $(s_{2k})_{k\in\mathbb{Z}}$ and $(n_k)_{k\in\mathbb{Z}}$ given above (Fig. 1).

A is a closed subset of $C(\mathbb{R})$, where $C(\mathbb{R})$ is endowed with the compact-open topology (that is, the topology of uniform convergence on compact subset of $\mathbb{R}$), therefore a complete separable metric space. Indeed, if $(f_j)_j$ is a sequence of functions in A that converges to certain $f \in C(\mathbb{R})$, then each f_j has associated sequences $(s_{2k}(j))_{k\in\mathbb{Z}}$ and $(n_k(j))_{k\in\mathbb{Z}}$ satisfying conditions (a)–(f) above. From the convergence with respect to the compact-open topology we deduce that there exist the limits $\lim_j s_{2k}(j)$ and $\lim_j n_k(j)$ for each $k \in \mathbb{Z}$. Now, we either have $\lim_j s_{-2}(j) < 0$, so that $s_{2k} := \lim_j s_{2k}(j)$ and $n_k := \lim_j n_k(j)$, $k \in \mathbb{Z}$, define the sequences that make $f \in A$, or we have $\lim_j s_{-2}(j) = 0$, that yields $s_{2k} := \lim_j s_{2k-2}(j)$ and $n_k := \lim_j n_{k-1}(j)$, $k \in \mathbb{Z}$, the defining sequence for $f \in A$. We will introduce a measure on A. Let us consider the Lebesgue measure λ on $\mathbb{R}$, and let p be the probability measure defined on $\mathbb{N}$ such that $p(\{j\}) = p_j$, with $0 < p_j < 1$, $p(\mathbb{N}) = \sum_{j=1}^{\infty} p_j = 1$ and, if

$$\beta_j := \left(\sum_{i=1}^{j} p_i\right)^{N_{j+1}-N_j}, \quad j \in \mathbb{N}, \quad \text{then} \quad \prod_{j=1}^{\infty} \beta_j > 0. \tag{2}$$

We define the map $\Psi : A \to (\mathbb{R}\times\mathbb{N})^{\mathbb{Z}}$ given by $\Psi(f_{(s_{2j}, n_j)_{j\in\mathbb{Z}}}) = (s_{2j}, n_j)_{j\in\mathbb{Z}}$. The map Ψ is continuous on $A\setminus A_0$, where

$$A_0 := \left\{ f = f_{(s_{2j}, n_j)_j} \in A \; ; \; s_0 = 0 \right\}.$$

Ψ is also continuous on A_0, thus Ψ is Borel measurable.

Let $\Pi_n : (\mathbb{R}\times\mathbb{N})^{\mathbb{Z}} \to (\mathbb{R}\times\mathbb{N})^{2n+1}$ be the projection onto the corresponding coordinate space centered at index 0 and define the measure $\widetilde{\lambda}_n = (\lambda \otimes p)^{2n+1}$ on $\Pi_n(\Psi(A))$. We have

$$\Pi_1(\Psi(A)) = \Big\{((s_{-2}, n_{-1}), (s_0, n_0), (s_2, n_1)) \in (\mathbb{R} \times \mathbb{N})^3;$$
$$s_0 \ge 0,\ s_{-2} < 0,\ \frac{1}{2} \le s_{2i} - s_{2i-2} \le \frac{3}{2},\ i = 0, 1\Big\},$$

and its associated measure is

$$\widetilde{\lambda}_1(\Pi_1(\Psi(A))) = \int_{-\frac{1}{2}}^{0} \left(\int_{s_{-2}+\frac{1}{2}}^{s_{-2}+\frac{3}{2}} \left(\int_{s_0+\frac{1}{2}}^{s_0+\frac{3}{2}} ds_2 \right) ds_0 \right) ds_{-2}$$
$$+ \int_{-\frac{3}{2}}^{-\frac{1}{2}} \left(\int_{0}^{s_{-2}+\frac{3}{2}} \left(\int_{s_0+\frac{1}{2}}^{s_0+\frac{3}{2}} ds_2 \right) ds_0 \right) ds_{-2} = \frac{1}{2} + \frac{1}{2} = 1.$$

Analogously, $\widetilde{\lambda}_n(\Pi_n(\Psi(A))) = 1$ for all $n \in \mathbb{N}$. Let $\mathfrak{B}_n$ be the σ-algebra of Borel subsets of $\Pi_n(\Psi(A))$. We consider $\mathfrak{A} := \bigcup_{n\in\mathbb{N}} \Pi_n^{-1}(\mathfrak{B}_n)$, which is an algebra consisting of Borel subsets of $\Psi(A)$ since $\Pi_n^{-1}(\mathfrak{B}_n) \subset \Pi_{n+1}^{-1}(\mathfrak{B}_{n+1})$ for all $n \in \mathbb{N}$. Also, the σ-algebra generated by $\mathfrak{A}$ coincides with the family $\widetilde{\mathfrak{B}}$ of all Borel subsets of $\Psi(A)$ because $\mathfrak{A}$ contains the open subsets of $\Psi(A)$. For each $n \in \mathbb{N}$ and $B \in \Pi_n^{-1}(\mathfrak{B}_n)$, we define $\widetilde{\lambda}(B) = \widetilde{\lambda}_n(\Pi_n(B))$. $\widetilde{\lambda}$ is a well-defined probability measure on $\mathfrak{A}$ with full support since $\widetilde{\lambda}_n(\Pi_n(B)) = \widetilde{\lambda}_{n+1}(\Pi_{n+1}(B))$ for every $B \in \Pi_n^{-1}(\mathfrak{B}_n)$, $n \in \mathbb{N}$. There is a unique extension of $\widetilde{\lambda}$ to $\widetilde{\mathfrak{B}}$, for which we keep the same notation (see, e.g., [29]). Now, since A is a complete separable metric space and $\Psi : A \to (\mathbb{R} \times \mathbb{N})^{\mathbb{Z}}$ is an injective measurable map, we have that the family of Borel sets of A equals $\Psi^{-1}(\widetilde{\mathfrak{B}})$ (see, e.g., [34, Corollary 3.3]), and we obtain that $\overline{\mu} := \widetilde{\lambda} \circ \Psi$ is a Borel probability measure on A with full support.

Moreover, A is R_t-invariant for any $t \in \mathbb{R}$, where $(R_t)_{t\in\mathbb{R}}$ is the translation C_0-group, since given $f_{(s_{2j},n_j)_j} \in A$ we have that $R_t(f_{(s_{2j},n_j)_j}) = f_{(t+s_{2j+2k},n_{j+k})_j} \in A$, where

$$k := \min\{j \in \mathbb{Z}\ ;\ t + s_{2j} \ge 0\}. \tag{3}$$

The definition of $\overline{\mu}$ easily yields that $\overline{\mu}$ is $(R_t)_{t\in\mathbb{R}}$-invariant.

We also note that $\overline{\mu}$ is strongly mixing with respect to the translation C_0-group $(R_t)_{t\in\mathbb{R}}$. Actually, it suffices to prove it on a basis of open sets of A.

Let us define, for each

$$\alpha = \left((s_{2j})_{j=-n}^m, (n_j)_{j=-n}^m, \varepsilon\right) \in \mathbb{R}^{n+m+1} \times \mathbb{N}^{n+m+1} \times]0, 1/4[$$

with $s_{-2n} < \ldots < s_{-2} < 0 \le s_0 < s_2 < \ldots < s_{2m}$, $\frac{1}{2} \le s_{2j+2} - s_{2j} \le \frac{3}{2}$, $j = -n, \ldots, m-1$, the set

$$A_\alpha = \Big\{ f \in A\ ;\ \exists\, t_{2j} \in]s_{2j} - \varepsilon, s_{2j} + \varepsilon[\text{ with } f(t_{2j}) = n_j,\ f(t_{2j+1}) = 0 \quad \text{for}$$
$$t_{2j+1} := \frac{t_{2j} + t_{2j+2}}{2},\ j = -n, \ldots, m-1,\ f''|_{]t_i, t_{i+1}[} \equiv 0,\ i = -2n, \ldots, 2m-1 \Big\}.$$

They form a basis of open sets in A as a topological subspace of $C(\mathbb{R})$. Let A_α and $A_{\alpha'}$ be two elements from the above basis, where

$$\alpha = \left((s_{2j})_{j=-n}^m, (n_j)_{j=-n}^m, \varepsilon\right) \qquad \text{and} \qquad \alpha' = \left((s'_{2j})_{j=-n'}^{m'}, (n'_j)_{j=-n'}^{m'}, \varepsilon'\right).$$

If t is large enough then $[s_{-n}-\varepsilon, s_m+\varepsilon]\bigcap[t+s'_{-n'}-\varepsilon', t+s'_{m'}+\varepsilon']=\emptyset$ and because of the definition of $\overline{\mu}$ and the empty intersection of the previous intervals we have:

$$\overline{\mu}(A_\alpha \cap R_t(A_{\alpha'})) = \overline{\mu}(A_\alpha)\overline{\mu}(A_{\alpha'}).$$

Let us consider the subset of A given by

$$H = \{f_{(s_{2k},n_k)_k} \in A\ ;\ n_k = f(s_{2k}) \in \{1,\dots,m\}\ \text{ if } N_m \le |k| < N_{m+1},\\ m \in \mathbb{N},\ f(s_{2k}) = 1 \text{ for } |k| < N_1\}.$$

Clearly, H is a closed subset A which is bounded in $C(\mathbb{R})$. An easy argument shows that the subsets of A that are bounded in $C(\mathbb{R})$ are relatively compact, thus H is compact. Indeed, if $(f_j)_j$ is a sequence in A that is bounded in $C(\mathbb{R})$, then by passing to a subsequence, if necessary, we may suppose that there exist the limits $\lim_j s_{2k}(j)$ and $\lim_j n_k(j)$, for each $k \in \mathbb{Z}$, where $(s_{2k}(j))_{k\in\mathbb{Z}}$ and $(n_k(j))_{k\in\mathbb{Z}}$ are the sequences associated with f_j satisfying the corresponding conditions (a)–(f) before, $j \in \mathbb{N}$. This implies that $(f_j)_j$ has a limit point $f \in A$.

Let $Z = \bigcup_{t\in\mathbb{R}} R_t(H) = \bigcup_{j\in\mathbb{Z}} R_j(H)$, therefore a Borel subset of A. We easily get

$$\overline{\mu}(Z) \ge \overline{\mu}(H) = (p_1)^{2N_1-1}\left(\prod_{l=1}^{\infty}\beta_l\right)^2 > 0.$$

The last inequality is obtained by using Fubini's theorem again.

Since Z is R_t-invariant and has positive measure, then $\overline{\mu}(Z) = 1$.

2. The map Φ.

We define the map $\Phi : Z \to X$ by

$$\Phi\left(f_{(s_{2j},n_j)_j}\right) = \sum_{j\le -2}\int_{s_{2j}}^{s_{2j+2}} S_{-t}x_{n_j}\,dt + \int_{s_{-2}}^{0} S_{-t}x_{n_{-1}}\,dt \\ + \int_0^{s_0} T_t x_{n_{-1}}\,dt + \sum_{j\ge 0}\int_{s_{2j}}^{s_{2j+2}} T_t x_{n_j}\,dt. \tag{4}$$

Φ is well defined since, given $f_{(s_{2j},n_j)_j} \in R_{t_0}(H)$, $t_0 \in \mathbb{R}$, and for $l \ge |t_0|$, we have that $n_k \le 2l$ if $N_l < |k| \le N_{l+1}$, which shows the convergence of the series in (4) by (1).

Let us see that $T_a \circ \Phi = \Phi \circ R_a$ for any $a > 0$. We will distinguish two cases:

Case 1 $s_{-2} < -a < 0$:

$$T_a \circ \Phi\left(f_{(s_{2j},n_j)_j}\right) = \sum_{j\le -2}\int_{a+s_{2j}}^{a+s_{2j+2}} S_{-t}x_{n_j} + \int_{a+s_{-2}}^{0} S_{-t}x_{n_{-1}} + \int_0^{a+s_0} T_t x_{n_{-1}} \\ + \sum_{j\ge 0}\int_{a+s_{2j}}^{a+s_{2j+2}} T_t x_{n_j} = \Phi\left(f_{(a+s_{2j},n_j)_j}\right) = \Phi \circ R_a\left(f_{(s_{2j},n_j)_j}\right)$$

since, in this case, $0 = \min\{j \in \mathbb{Z}\ ;\ a + s_{2j} \ge 0\}$.

Case 2 $s_{2k} < -a \leq s_{2k+2}$, for some $k \in \mathbb{Z}^-$, $k \leq -2$:

$$T_a \circ \Phi\left(f_{(s_{2j},n_j)_j}\right) = \sum_{j<k} \int\limits_{a+s_{2j}}^{a+s_{2j+2}} S_{-t}x_{n_j} + \int\limits_{a+s_{2k}}^{0} S_{-t}x_{n_k} + \int\limits_{0}^{a+s_{2k+2}} T_t x_{n_k}$$

$$+\sum_{j>k} \int\limits_{a+s_{2j}}^{a+s_{2j+2}} T_t x_{n_j} = \Phi\left(f_{(a+s_{2j+2k+2},n_{j+k+1})_j}\right) = \Phi \circ R_a\left(f_{(s_{2j},n_j)_j}\right)$$

since, in this case, $k+1 = \min\{j \in \mathbb{Z}\,;\ a+s_{2j} \geq 0\}$.

Also, Φ is continuous almost everywhere on $R_{t_0}(H)$ for any $t_0 \in \mathbb{R}$. Indeed, let $\left(f_{(s^k_{2j},n^k_j)_j}\right)_k$ be a sequence in $R_{t_0}(H)$ that converges to $f_{(s_{2j},n_j)_j} \in R_{t_0}(H)$ with $s_0 > 0$. Then, for any compact set $C \subset \mathbb{R}$, we have that

$$\lim_{k\to\infty} \sup_{x\in C} \left| f_{\left(s^k_{2j},n^k_j\right)_j}(x) - f_{(s_{2j},n_j)_j}(x) \right| = 0.$$

In particular, for any $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that,

$$\text{if } |j| \leq N \text{ and } k \geq k_0, \text{ then } n^k_j = n_j \text{ and } |s^k_{2j} - s_{2j}| < \varepsilon. \tag{5}$$

Fix $n > |t_0|$ and $N = N_n$. Let $\varepsilon > 0$ such that $\|\int_I S_{-t}x_{n_j}dt\| + \|\int_J T_t x_{n_j}dt\| < (3(N+1)2^{n+1})^{-1}$ whenever $I \subset]-\infty, 0]$ and $J \subset [0, +\infty[$ are intervals of length less than ε and $|j| \leq N$. By (5) and (1), there exists an integer k_0 such that for every $k \geq k_0$,

$$\begin{aligned}
&\left\| \Phi\left(f^k_{(s^k_{2j},n^k_j)_j}\right) - \Phi\left(f_{(s_{2j},n_j)}\right) \right\| \\
&\leq \left\| \sum_{j<-N_n} \int\limits_{s^k_{2j}}^{s^k_{2j+2}} S_{-t}x^k_{n_j} \right\| + \left\| \sum_{j>N_n} \int\limits_{s^k_{2j}}^{s^k_{2j+2}} T_t x^k_{n_j} \right\| \\
&\quad + \left\| \sum_{j<-N_n} \int\limits_{s_{2j}}^{s_{2j+2}} S_{-t}x_{n_j} \right\| + \left\| \sum_{j>N_n} \int\limits_{s_{2j}}^{s_{2j+2}} T_t x_{n_j} \right\| \\
&\quad + \sum_{-N_n\leq j\leq -2} \left\| \int\limits_{\min(s^k_{2j},s_{2j})}^{\max(s^k_{2j},s_{2j})} S_{-t}x_{n_{2j}} \right\| + \sum_{-N_n\leq j\leq -2} \left\| \int\limits_{\min(s^k_{2j+2},s_{2j+2})}^{\max(s^k_{2j+2},s_{2j+2})} S_{-t}x_{n_j} \right\| \\
&\quad + \left\| \int\limits_{\min(s^k_{-2},s_{-2})}^{\max(s^k_{-2},s_{-2})} S_{-t}x_{n_{-1}} \right\| + \left\| \int\limits_{\min(s^k_0,s_0)}^{\max(s^k_0,s_0)} T_t x_{n_{-1}} \right\| \\
&\quad + \sum_{0\leq j\leq N_n} \left\| \int\limits_{\min(s^k_{2j},s_{2j})}^{\max(s^k_{2j},s_{2j})} T_t x_{n_j} \right\| + \sum_{0\leq j\leq N_n} \left\| \int\limits_{\min(s^k_{2j+2},s_{2j+2})}^{\max(s^k_{2j+2},s_{2j+2})} T_t x_{n_j} \right\| < \frac{1}{2^{n-1}} + \frac{1}{2^n}.
\end{aligned}$$

This shows the continuity almost everywhere of $\Phi : R_t(H) \to X$ for every $t \in \mathbb{R}$. The map Φ is well-defined on Z, and $\Phi : Z \to X$ is Borel measurable.

3. The measure μ on X.
$L(s) := \Phi(R_s(H))$ is a countable union of compact sets in X for each $s \in \mathbb{R}$. Indeed,

$$\Phi(R_s(H)) = \Phi\left(\{f_{(s_{2j},n_j)} \in R_s(H)\ ;\ s_0 = 0\}\right) \cup \bigcup_{n\in\mathbb{N}} \Phi\left(\{f_{(s_{2j},n_j)} \in R_s(H)\ ;\ s_0 \geq 1/n\}\right).$$

We then have that $Y := \bigcup_{s\in\mathbb{R}} L(s) = \bigcup_{n\in\mathbb{N}} L(n)$ is a countable union of compact sets, thus a T_t-invariant Borel subset of X because $\Phi \circ R_t = T_t \circ \Phi$, $t > 0$.

We then define in X the measure $\mu(B) = \overline{\mu}(\Phi^{-1}(B))$ for all $B \in \mathfrak{B}(X)$. Obviously, μ is well-defined and it is a $(T_t)_t$-invariant strongly mixing Borel probability measure. The proof is completed by showing that μ has full support. In the proof of [32, Theorem 2.2] it was shown that, for $u_k := \int_0^1 T_t x_k dt$, $k \in \mathbb{N}$, the set $\{u_k\ ;\ k \in \mathbb{N}\}$ is dense in X. Thus, given a non-empty open set U in X, we pick $n \in \mathbb{N}$ and $\varepsilon > 0$ satisfying

$$\int_{s_0}^{s_2} T_t x_n\, dt + B(0, 1/2^n) \subset U$$

for any $s_0 \in [0, \varepsilon]$, $s_2 \in [1, 1+\varepsilon]$. Together with (1), this implies

$$\mu(U) \geq \mu\left(\left\{\Phi(f_{(s_{2j},n_j)_j})\ ;\ f_{(s_{2j},n_j)_j} \in Z,\ s_0 \in [0,\varepsilon],\ s_2 \in [1, 1+\varepsilon],\right.\right.$$

$$\left. n_0 = n,\ n_k = 1 \text{ if } 0 < |k| \leq N_n,\ n_k \leq 2l,\ \text{ for } N_l < |k| \leq N_{l+1},\ l \geq n\}\right)$$

$$\geq \varepsilon^2 p_n (p_1)^{2N_n} \prod_{l=n}^{\infty} \left(\prod_{N_l < |k| \leq N_{l+1}} \sum_{r=1}^{2l} p_r \right) > \varepsilon^2 p_n (p_1)^{2N_n} \prod_{l=n}^{\infty} (\beta_l)^2 > 0.$$

□

Remark 1 There exists an alternative way of defining the measure on the space of continuous functions, by using Brownian motions (for more details see [35,36]). We denote by $\mathfrak{B} = \mathfrak{B}(C([0,\infty)))$, the σ-algebra of Borel subsets of $C([0,\infty))$. Let ω_t, $t \geq 0$, be a Brownian motion defined on a probability space (Ω, Σ, P). Assume that the sample functions of ω_t are continuous. Setting $\xi_t = e^t \omega_{e^{-2t}}$ for $t \geq 0$, then ξ_t is a stationary Gaussian process with mean value $E\xi_t = 0$ and correlation function $E\xi_t\xi_{t+h} = e^{-|h|}$. Then the measure on $\mathfrak{B} = \mathfrak{B}(C([0,\infty)))$ induced by ξ_t is strongly mixing with full support. The details of the construction of the measure can be found in [36].

In [32, Cor. 2.3] some conditions, expressed in terms of eigenvector fields for the infinitesimal generator of the C_0-semigroup, were given which ensure that the assumptions of Theorem 3 are satisfied. In consequence we also obtain the stronger result of existence of invariant strongly mixing measures under the same conditions. A different argument for the existence of invariant strongly mixing measures for C_0-semigroups has been obtained in [14] under weaker assumptions on the eigenvector fields for the generator.

Corollary 1 *Let X be a separable complex Banach space and let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator A. Assume that there exists a family $(f_j)_{j\in\Gamma}$ of locally bounded measurable maps $f_j : I_j \to X$ such that I_j is an interval in $\mathbb{R}$, $f_j(I_j) \subset D(A)$, where $D(A)$ denotes the domain of the generator, $Af_j(t) = itf_j(t)$ for every $t \in I_j$, $j \in \Gamma$ and* span$\{f_j(t)\ :\ j \in \Gamma,\ t \in I_j\}$ *is dense in X. If either*

a) $f_j \in C^2(I_j, X)$, $j \in \Gamma$,
or
b) *X does not contain c_0 and $\langle \varphi, f_j \rangle \in C^1(I_j)$, $\varphi \in X'$, $j \in \Gamma$, then there is a $(T_t)_{t \geq 0}$-invariant strongly mixing Borel probability measure μ on X with full support.*

3 Applications

In this section we will present several applications of the previous results to the (chaotic) behaviour of the solution C_0-semigroup to certain linear partial differential equations and infinite systems of linear differential equations.

Example 1 Let us consider the following linear perturbation of the one-dimensional Ornstein–Uhlenbeck operator

$$\mathcal{A}_\alpha u = u'' + bxu' + \alpha u,$$

where $\alpha \in \mathbb{R}$, with domain

$$D(\mathcal{A}_\alpha) = \left\{ u \in L^2(\mathbb{R}) \cap W^{2,2}_{\text{loc}}(\mathbb{R});\ \mathcal{A}_\alpha u \in L^2(\mathbb{R}) \right\}.$$

We know that, if $\alpha > b/2 > 0$, then the semigroup generated by $\mathcal{A}_\alpha$ in $L^2(\mathbb{R})$ is chaotic [18] and frequently hypercyclic [32]. Actually, it was shown that the C_0-semigroup satisfies the hypothesis of Corollary 1 [32]. Therefore, we also obtain that it admits an invariant strongly mixing measure with full support.

Example 2 Rudnicki [38] recently showed the existence of invariant strongly mixing measures for some C_0-semigroups generated by a partial differential equation of population dynamics. More precisely, he reduced the equation to

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = au(t,x) + bu(t,2x),$$

whose formal solution, given the initial condition $u(0,x) = u_0(x)$, is

$$u(t,x) := e^{at} \sum_{n=0}^{\infty} \frac{(bt)^n}{n!} u_0(2^n e^{-t} x).$$

He considered the space

$$X = X_{\alpha,\beta} := \left\{ u \in C(]0,\infty[);\ \lim_{x\to 0} x^\alpha |u(x)| = 0,\ \lim_{x\to\infty} x^\beta |u(x)| = 0 \right\}$$

endowed with the norm $\|u\| := \sup_{x\in]0,\infty[} |u(x)|\rho(x)$, where $\rho(x) = x^\alpha$ if $x \leq 1$ and $\rho(x) = x^\beta$ if $x > 1$. If $2^a b \log 2 < e^{-1}$, $\beta < \log_2 b + \log_2(\log 2)$, and $\alpha > \alpha_0$, where α_0 satisfies $(a + \alpha_0)2^{\alpha_0} = b$, then there exists a Borel strongly mixing probability measure μ on X with full support which is invariant under the solution C_0-semigroup generated by the above equation [38, Thm 1]. Actually, this fact was shown by reducing the problem to the translation flow $(R_t)_{t\in\mathbb{R}}$ on the space

$$Y := \left\{ g \in C(\mathbb{R});\ \lim_{|x|\to\infty} \frac{g(x)}{x} = 0 \right\},$$

of weighted continuous functions with the norm

$$\|g\|_Y = \sup_{x\in\mathbb{R}} \frac{|g(x)|}{1+|x|}.$$

The corresponding generator is $A = D$, the derivative operator. We can apply directly our Corollary 1 to the map $f : \mathbb{R} \to Y$ given by $[f(t)](x) := e^{itx}$, which is a C^2-map, and obtain the same result since $\operatorname{span}\{f(t);\ t \in \mathbb{R}\}$ is the set of trigonometric polynomials, which is dense in Y.

Example 3 The chaotic behaviour associated to birth-and-death processes has been widely studied by Banasiak et al [4–6,8]. We will consider three cases that are shown to admit invariant strongly mixing measures.

1. In [8], Banasiak and Moszyński studied the following "birth-and-death" model with constant coefficients:

$$\begin{aligned}\frac{df_1}{dt} &= (\mathcal{L}f)_1 = af_1 + df_2,\\ \frac{df_n}{dt} &= (\mathcal{L}f)_n = bf_{n-1} + af_n + df_{n+1}, \qquad n \geq 2.\end{aligned} \tag{6}$$

   Among other things, they studied the chaotic behaviour of the solution C_0-semigroup.

Theorem 4 ([8]) *Let $a,\ b,\ d \in \mathbb{R}$ satisfy $0 < |b| < |d|$ and $|a| < |b+d|$. Then the solution C_0-semigroup to the Cauchy problem (6) is chaotic on ℓ^p.*

Actually, to show this result they used a spectral criterion (see [7] and [22]) which is less general than the criterion of Corollary 1. In consequence, we obtain that the solution C_0-semigroup to the Cauchy problem (6) admits an invariant strongly mixing measure on ℓ^p with full support.

2. A more general model was studied by Banasiak et al.[6],

$$\begin{aligned}\frac{df_1}{dt} &= a_1 f_1 + d_1 f_2,\\ \frac{df_n}{dt} &= b_n f_{n-1} + a_n f_n + d_n f_{n+1}, \qquad n \geq 2.\end{aligned} \tag{7}$$

   with $a_n, b_n, d_n \in \mathbb{R}$ and the infinite matrix

$$\mathcal{L} = \begin{pmatrix} a_1 & d_1 & & & \\ b_2 & a_2 & d_2 & & \\ & b_3 & a_3 & d_3 & \\ & & b_4 & a_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

They intended to obtain sub-chaos (i.e., chaos on a subspace) results for birth-and-death type models with proliferation under certain conditions on the coefficients. In [2], Aroza and Peris studied the same model with more general coefficients and considered the Banach space X on which the operator associated with $\mathcal{L}$ generates a C_0-semigroup. Given $1 \leq p < \infty$, let

$$X = X(\gamma) := \left\{ f \in \ell^p : \mathcal{L}^n f \in \ell^p, \forall n \in \mathbb{N},\ \text{and } ||f|| := \sum_{n=0}^{\infty} ||\mathcal{L}^n f||_p \gamma^{-n} < \infty \right\}.$$

If the sequences $(a_n)_n$, $(b_n)_n$ and $(d_n)_n$ are bounded, $\mathcal{L}$ has an associated bounded operator $\mathcal{S}_p$ on ℓ^p, with spectral radius $r(S_p) < \infty$, and $X(\gamma) = \ell^p$ for $\gamma > r(\mathcal{S}_p)$. If any of the sequences $(a_n)_n$, $(b_n)_n$ and $(d_n)_n$ is unbounded, we have that the operator $\mathcal{S}_X$ associated with $\mathcal{L}$ is a bounded operator on X and, therefore, it generates a C_0-semigroup $\mathcal{T}_X$ on X. They obtained the following result:

Theorem 5 ([2]) *Let (a_n), (b_n) and $(d_n)_n$ be sequences of real numbers such that $d_n \neq 0$ for all $n \in \mathbb{N}$, $1 \leq p < \infty$, and $\gamma > 0$. Assume that either*

1. $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} d_n = d \neq 0$ *with* $|b| < |d|$ *and* $|a| < |b+d|$ *or*
2. $\lim_{n\to\infty} \frac{a_n}{d_n} = \alpha$, $\lim_{n\to\infty} \frac{b_n}{d_n} = \beta$, $\lim_{n\to\infty} d_n = \infty$ *with* $\alpha^2 \neq 4\beta$, $|\beta| < 1$ *and* $|\alpha| < |1+\beta|$

then the C_0-semigroup $\mathcal{T}_X$ is sub-chaotic on $X(\gamma)$. Moreover, in case 1, $\mathcal{S}_p$ generates a sub-chaotic C_0-semigroup $\mathcal{T}_p$ on ℓ^p.

Actually, to show this result they proved that the solution C_0-semigroup satisfies the spectral criterion of [7], in particular the conditions of Corollary 1 on a certain subspace Y. Thus, we obtain that the corresponding solution C_0-semigroup admits an invariant strongly mixing measure μ on $X(\gamma)$ whose support is Y.

3. Let us consider the death model

$$\begin{cases} \frac{\partial f_n}{\partial t} = -\alpha_n f_n + \beta_n f_{n+1}, & n \geq 1, \\ f_n(0) = a_n, & n \geq 1 \end{cases} \tag{8}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are bounded positive sequences and $(a_n)_n \in \ell^1$ is a real sequence. Considering $X = \ell^1$, and the map A given by

$$Af = (-\alpha_n f_n + \beta_n f_{n+1})_n \text{ for } f = (f_n)_n \in X,$$

since A is a bounded operator on ℓ^1, it generates a C_0-semigroup $(T_t)_{t\geq 0}$ which is solution of (8). It is shown in [28], that if

$$\sup_{n\geq 1} \alpha_n < \liminf_{n\to\infty} \beta_n$$

then the semigroup $(T_t)_{t\geq 0}$ satisfies the hypothesis of the spectral criterion [22], and then we can ensure the existence of an invariant strongly mixing measure with full support on X.

Example 4 Let us consider the solution semigroup $(e^{tA})_{t\geq 0}$ of the hyperbolic heat transfer equation problem:

$$\begin{cases} \tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \\ u(0,x) = \varphi_1(x), \quad x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x) = \varphi_2(x), \quad x \in \mathbb{R} \end{cases} \tag{9}$$

where φ_1 and φ_2 represent the initial temperature and the initial variation of temperature, respectively, $\alpha > 0$ is the thermal diffusivity, and $\tau > 0$ is the thermal relaxation time. We can represent it as a C_0-semigroup on the product of a certain function space with itself. We set $u_1 = u$ and $u_2 = \frac{\partial u}{\partial t}$. Then the associated first-order equation is:

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau}I \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R} \end{cases} \tag{10}$$

We fix $\rho > 0$ and consider the space

$$X_\rho = \left\{ f : \mathbb{R} \to \mathbb{C};\ f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n,\ (a_n)_{n\geq 0} \in c_0 \right\}$$

endowed with the norm $||f|| = \sup_{n\geq 0} |a_n|$.

Since

$$A := \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau}I \end{pmatrix}. \tag{11}$$

is an operator on $X := X_\rho \oplus X_\rho$, we have that $(e^{tA})_{t\geq 0}$ is the C_0-semigroup solution of 9. We know from [21] and [28] that, given α, τ and ρ such that $\alpha\tau\rho > 2$, the solution semigroup $(e^{tA})_{t\geq 0}$ defined on $X_\rho \oplus X_\rho$ satisfies the hypothesis of the spectral criterion [22], and we conclude the existence of an invariant strongly mixing measure with full support on $X_\rho \oplus X_\rho$.

Example 5 In [15], Black and Scholes proved that under some assumptions about the market, the value of a stock option $u(x,t)$, as a function of the current value of the underlying asset $x \in \mathbb{R}^+ = [0,\infty)$ and time, satisfies the final value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru & \text{in } \mathbb{R}^+ \times [0,T] \\ u(0,T) = 0 & \text{for } t \in [0,T] \\ u(x,T) = (x-p)^+ & \text{for } x \in \mathbb{R}^+ \end{cases}$$

where $p > 0$ represents a given strike price , $\sigma > 0$ is the volatility and $r > 0$ is the interest rate. Let $v(x,t) = u(x, T-t)$, then it satisfies the forward Black–Scholes equation, defined for all time $t \in \mathbb{R}^+$ by

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv & \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \\ v(0,T) = 0 & \text{for } t \in \mathbb{R}^+ \\ v(x,0) = f(x) & \text{for } x \in \mathbb{R}^+ \end{cases}$$

with

$$f(x) = (x-p)^+ = \begin{cases} x-p & \text{if } x > p \\ 0 & \text{if } x \leq p. \end{cases}$$

In order to express this problem in an abstract form , we define $D_\nu = \nu x \frac{\partial}{\partial x}$, where $\nu = \frac{\sigma}{\sqrt{2}}$ and $\mathcal{B} = (D_\nu)^2 + \gamma(D_\nu) - rI$, with $\gamma = \frac{r}{\nu} - \nu$. Then the problem can be reformulated as:

$$\begin{cases} \frac{\partial v}{\partial t} = \mathcal{B}v, \\ v(0,T) = 0, \\ v(x,0) = f(x) \quad \text{for } x \in \mathbb{R}^+. \end{cases}$$

Recently in [26], a simple explicit representation of the solution of the Black–Scholes equation has been given and this representation holds in the spaces $Y^{s,\tau}$. Let

$$Y^{s,\tau} = \left\{ u \in C((0,\infty))\ ;\ \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0, \quad \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0 \right\}$$

be endowed with the norm

$$||u||_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right|.$$

It is shown that the C_0-semigroup solution of the Black–Scholes equation can be represented by $T_t := f(tD_\nu)$, where

$$f(z) = e^{g(z)} \text{ with } g(z) = z^2 + \gamma z - r \text{ and } D_\nu = \nu x \frac{\partial}{\partial x}.$$

For more information and details see [15].

In [24], it is proved that the Black–Scholes semigroup is strongly continuous and chaotic for $s > 1$, $\tau \geq 0$ with $s\nu > 1$. We will see that, with a little more work, the Black–Scholes semigroup satisfies the spectral criterion in [22] under the same restrictions on the parameters and, therefore, the hypothesis of Corollary 1.

Let $s > \frac{1}{\nu}$, $0 < \nu < 1$ and $s > 1$. Let $S_s = \{\lambda \in \mathbb{C}\ ;\ 0 < Re\lambda < s\nu\}$. By Lemma 3.5 in [24], we have that $g(S_s) \cap i\mathbb{R} \neq \emptyset$. Then there exists an open ball $U \subset g(S_s)$ such that $U \cap i\mathbb{R} \neq \emptyset$ and such that $U \cap \mathbb{R} = \emptyset$. In particular, we find an inverse g^{-1} well defined (and holomorphic) on U. We set $F = L \circ g^{-1}$, $F : U \to Y^{s,\tau}$, where $L : S_s \to Y^{s,\tau}$ is defined as $L(\lambda) = h_{\frac{\lambda}{\nu}}$, with $h_\lambda(x) = x^\lambda$. It is clear that F is weakly holomorphic since L is weakly holomorphic [24]. Finally, $AF(\lambda) = g(\nu \frac{g^{-1}(\lambda)}{\nu})F(\lambda) = \lambda F(\lambda)$ for any $\lambda \in U$, where $(A, D(A))$ is the generator of the Black–Scholes semigroup, and the equality $\langle F(\lambda), \psi\rangle = 0$ for a fixed $\psi \in (Y^{s,\tau})^*$ and for every $\lambda \in U$ necessarily implies $\psi = 0$ [24, Thm 3.6]. Thus, the spectral criterion in [22] is satisfied and the Black–Scholes semigroup admits an invariant strongly mixing Borel probability measure on $Y^{s,\tau}$ with full support by Corollary 1.

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